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Some integrals involving Legendre polynomials and associated Legendre functions

L D Salem† and H S Wio‡

Centro Atómico Bariloche§ and Instituto Balseiro||, 8400 SC de Bariloche, RN, Argentina

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Abstract. The definite integrals $\int_{-1}^{1} P_l(x)\, P_\lambda^\mu(x)\, dx$ are calculated by explicitly writing the Legendre polynomials and their associated Legendre functions in a suitable form. Selection rules derived from this approach are obtained.

1. Introduction

When studying the linear response of an axially symmetric finite system (such as an atomic nucleus) under the action of an external field with a definite multipolarity, say $(\lambda, \pm\mu)$, the need for evaluating integrals involving Legendre polynomials and their associated Legendre functions arises [1], i.e.

$$I(l, \lambda, \mu) = \int_{-1}^{1} dx\, P_l(x) P_\lambda^\mu(x) \qquad l = 0, 1, 2, \ldots;\ 0 \leqslant \mu \leqslant \lambda = 0, 1, 2, \ldots \tag{1.1}$$

where $P_\lambda(x)$ and $P_\lambda^\mu(x)$ are defined by [2]

$$P_l(x) = \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l} (1 - x^2)^l \qquad l = 0, 1, 2, \ldots \tag{1.2}$$

$$P_\lambda^\mu(x) = (-1)^\mu (1 - x^2)^{\mu/2} \frac{d^\mu}{dx^\mu} P_\lambda(x) \qquad 0 \leqslant \mu \leqslant \lambda = 0, 1, \ldots. \tag{1.3}$$

Surprisingly, such integrals are not found in the literature [2-5], with the trivial exception of the case $\mu = 0$ where the orthogonality relation between Legendre polynomials can be invoked:

$$I(l, \lambda, 0) = \int_{-1}^{1} P_l(x) P_\lambda(x)\, dx = \frac{2}{2l + 1} \delta_{l,\lambda} \qquad l, \lambda = 0, 1, 2, \ldots. \tag{1.4}$$

However, selection rules, which tell us immediately when the integral (1.1) vanishes, can be obtained. The first one is related to the parity of the integrand and this results

† Fellow of CONICET, Argentina.
‡ Member of CONICET, Argentina.
§ Comisión Nacional de Energía Atómica (CNEA).
|| CNEA and Universidad Nacional de Cuyo.

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$$I(l, \lambda, \mu) = 0 \qquad \text{when } l+\lambda-\mu \text{ is odd.} \tag{1.5}$$

Another selection rule valid for even values of μ arises from the orthogonality of the $P_l(x)$ to all polynomials of degree less than l, which gives

$$I(l, \lambda, \mu) = 0 \qquad \text{when } l>\lambda \text{ and } \mu = 2M. \tag{1.6}$$

This paper is organised as follows. In § 2 we derive the selection rules (1.5) and (1.6), and work out (1.1) when μ is even. In § 3 we proceed for the case μ odd by writing the integrand of (1.1) as $(1-x^2)^{1/2}$ times a polynomial in x. Finally, in § 4, we summarise the results and present explicit examples in order to show the general trend.

2. Selection rules and case μ even

According to (1.2) and (1.3) we have explicitly, for $P_l(x)$ and $P_\lambda^\mu(x)$,

$$P_l(x) = \sum_{i=0}^{L} a_i x^{l-2L+2i} \qquad l = 0, 1, 2, \ldots \tag{2.1}$$

$$P_\lambda^\mu(x) = \sum_{j=0}^{\nu} \alpha_j x^{\lambda-\mu-2\nu+2j}(1-x^2)^{\mu/2} \qquad 0 \leq \mu \leq \lambda,\ \lambda = 0, 1, \ldots \tag{2.2}$$

where $L = [l/2]$ and $\nu = [(\lambda-\mu)/2]$ ($[x]$ stands for the integer part of x), and the coefficients a_i and α_j are given by

$$a_i = \frac{(-1)^{i-L}}{2^l} \frac{(2l-2L+2i)!}{(L-i)!(l-L+i)!(l-2L+2i)!} \qquad i = 0, 1, \ldots, L \tag{2.3}$$

$$\alpha_j = \frac{(-1)^{\mu-\nu+j}}{2^\lambda} \frac{(2\lambda-2\nu+2j)!}{(\nu-j)!(\lambda-\nu+j)!(\lambda-\mu-2\nu+2j)!} \qquad j = 0, 1, \ldots, \nu. \tag{2.4}$$

From (2.1) and (2.2) we find the parity of the integrand in (1.1) to be $(-1)^{l+\lambda-\mu}$ and this results in the first selection rule expressed by (1.5). We remark that this selection rule is valid for the whole range of l, λ and μ values.

The associated Legendre function $P_\lambda^\mu(x)$ can always be written as

$$P_\lambda^\mu(x) = F_\lambda^\mu(x)(1-x^2)^{\mu/2-M} \tag{2.5}$$

where $M = [\mu/2]$ and $F_\lambda^\mu(x)$ is a polynomial of degree λ when μ is even and $\lambda-1$ when μ is odd. Explicitly,

$$\begin{aligned} F_\lambda^\mu(x) &= \left(\sum_{j=0}^{\nu} \alpha_j x^{2j+\lambda-\mu-2\nu}\right)\left(\sum_{k=0}^{M} \beta_k x^{2k}\right) \\ &= \sum_{q=0}^{M+\nu} b_q x^{2q+\lambda-\mu-2\nu} \end{aligned} \tag{2.6}$$

where the coefficients β_k and b_q are given by

$$\beta_k = (-1)^{-k}\binom{M}{k} \qquad k = 0, 1, \ldots, M \tag{2.7}$$

$$b_q = \sum_{j=\max\{0,\, q-M\}}^{\min\{q,\, \nu\}} \alpha_j \beta_{q-j}$$

$$= \frac{(-1)^{\mu-\nu-q}}{2^{\lambda}} \sum_{j=\max\{0,\, q-M\}}^{\min\{q,\, \nu\}} \times \frac{(2\lambda-2\nu+2j)!}{(\nu-j)!(\lambda-\nu+j)!(\lambda-\mu-2\nu+2j)!} \binom{M}{q-j}$$

$$q = 0, 1, \ldots, M+\nu. \tag{2.8}$$

The parity of $F_\lambda^\mu(x)$ is that of $P_\lambda^\mu(x)$, say $(-1)^{\lambda \pm \mu}$, as can be seen from (2.6).

In particular, when $\mu = 2M$ we have $\nu = [\lambda/2] - M \equiv \Lambda - M$ which yields

$$P_\lambda^\mu(x) = F_\lambda^\mu(x) = \sum_{q=0}^{\Lambda} b_q x^{2q+\lambda-2\Lambda}. \tag{2.9}$$

Inserting this expression in (1.1) we have

$$I(l, \lambda, \mu = 2M) = \sum_{q=0}^{\Lambda} b_q \int_{-1}^{1} x^{2q+\lambda-2\Lambda} P_l(x)\, dx. \tag{2.10}$$

With the aid of the tabulated integrals [2, 4] of the form

$$\int_{-1}^{1} x^s P_l(x)\, dx = \begin{cases} 0 & \text{when } s+l \text{ is odd or } s<l \\ \dfrac{[\frac{1}{2}(s+l)]!\, s!\, 2^{l+1}}{[\frac{1}{2}(s-l)]!\,(s+l+1)!} & \text{when } s+l \text{ is even and } s \geq l \end{cases}$$

and replacing this result in (2.10), the selection rule given in (1.6) follows. Moreover, for $l \leq \lambda$ and $l+\lambda$ odd the integral of (1.1) vanishes as each term in (2.10) does; this is also in accord with the selection rule (1.5).

Meanwhile for $l \leq \lambda$ and $l+\lambda$ even, we have

$$I(l, \lambda, \mu = 2M)$$

$$= 2^{l+1-\lambda}(-1)^{\Lambda-\mu/2} \sum_{q=(l-\lambda)/2+\Lambda}^{\Lambda} (-1)^q \times \frac{[q-\Lambda+(\lambda+l)/2]!\,(2q+\lambda-2\Lambda)!}{[q-\Lambda+(\lambda-l)/2]!\,(2q+l+\lambda-2\Lambda+1)!}$$

$$\times \sum_{j=\max\{0,\, q-\mu/2\}}^{\min\{q,\, \Lambda-\mu/2\}} \binom{\mu/2}{q-j} \frac{(2\lambda-2\Lambda+\mu+2j)!}{(\Lambda-\mu/2-j)!\,(\lambda+\Lambda+\mu/2+j)!\,(\lambda-2\Lambda+2j)!}. \tag{2.11}$$

For the particular case $\mu = 0$ this equation verifies the orthogonality relation (1.4).

Just for the sake for clarity we explicitly present the results given above for the two possible cases left by the parity selection rule.

(*a*) When $l = 2L$ and $\lambda = 2\Lambda$ we have

$$I(l = 2L, \lambda = 2\Lambda, \mu = 2M)$$

$$= \frac{(-1)^{(\mu-\lambda)/2}}{2^{\lambda-l-1}} \sum_{q=l/2}^{\lambda/2} (-1)^q \frac{(q+l/2)!\,(2q)!}{(q-l/2)!\,(2q+l+1)!}$$

$$\times \sum_{j=\max\{0,\, q-\mu/2\}}^{\min\{q,\, (\lambda-\mu)/2\}} \binom{\mu/2}{q-j} \frac{(\lambda+\mu+2j)!}{[(\lambda-\mu)/2-j]!\,[(\lambda+\mu)/2+j]!\,(2j)!}. \tag{2.12}$$

(*b*) When $l=2L+1$ and $\lambda=2\Lambda+1$ we have

$$I(l=2L+1, \lambda=2\Lambda+1, \mu=2M)$$

$$=\frac{(-1)^{(\lambda-\mu+1)/2}}{2^{\lambda-l-1}}\sum_{q=(l-1)/2}^{(\lambda-1)/2}(-1)^{-q}\frac{[q+(1+l)/2]!(2q+1)!}{[q+(1-l)/2]!(2q+l+2)!}$$

$$\times\sum_{j=\max\{0,q-\mu/2\}}^{\min\{q,(\lambda-\mu-1)/2\}}\binom{\mu/2}{q-j}\frac{(\lambda+\mu+2j+1)!}{[(\lambda-\mu-1)/2-j]![(\lambda+\mu+1)/2+j]!(2j+1)!}. \tag{2.13}$$

3. Case μ odd

When μ is odd (i.e. $\mu=2M+1$) we can still integrate (1.1) by expanding the integrand as $(1-x^2)^{1/2}$ times a polynomial (see (2.5)). Moreover, according to the selection rule (1.5), for those values of l and λ such that $l+\lambda$ is an even number, the integral vanishes. Thus we concentrate on the calculation of (1.1) when $l+\lambda$ is odd; using (2.1) and (2.9) we get

$$I(l, \lambda, \mu=2M+1)$$

$$=\sum_{i=0}^{L}a_i\sum_{q=0}^{M+\nu}b_q\int_{-1}^{1}\mathrm{d}x(1-x^2)^{1/2}x^{l+\lambda-\mu+2(i+q-L-\nu)}$$

$$=\sum_{r=0}^{L+M+\nu}c_r\int_{-1}^{1}\mathrm{d}x(1-x^2)^{1/2}x^{l+\lambda-\mu+2(r-L-\nu)} \tag{3.1}$$

where

$$c_r=\sum_{i=\max\{0,r-(M+\nu)\}}^{\min\{r,L\}}a_ib_{r-i}$$

$$=\frac{(-1)^{\mu-\nu-L-r}}{2^{l+\lambda}}\sum_{i=\max\{0,r-(M+\nu)\}}^{\min\{r,L\}}\frac{(2l-2L+2i)!}{(L-i)!(l-L+i)!(l-2L+2i)!}$$

$$\times\sum_{j=\max\{0,r-i-M\}}^{\min\{r-i,\nu\}}\frac{(2\lambda-2\nu+2j)!}{(\nu-j)!(\lambda-\nu+j)!(\lambda-\mu-2\nu+2j)!}\binom{M}{r-i-j}$$

$$r=0,1,\ldots,L+M+\nu. \tag{3.2}$$

In (3.1) we find the tabulated [4] integrals of the form

$$\int_{-1}^{1}\mathrm{d}x\,x^{2s}(1-x^2)^{1/2}=\frac{\pi(2s)!}{2^{2s+1}s!(s+1)!}. \tag{3.3}$$

Replacing these results in (3.1) we obtain

$$I(l, \lambda, \mu=2M+1)$$

$$=\frac{(-1)^{\mu-\nu-L}\pi}{2^{l+\lambda+1}}\sum_{r=0}^{L+M+\nu}\frac{(-1)^{-r}(2r)!}{2^{2r}r!(r+1)!}$$

$$\times\sum_{i=\max\{0,r-(M+\nu)\}}^{\min\{r,L\}}\frac{(2l-2L+2i)!}{(L-i)!(l-L+i)!(l-2L+2i)!}$$

$$\times\sum_{j=\max\{0,r-i-M\}}^{\min\{r-i,\nu\}}\frac{(2\lambda-2\nu+2j)!}{(\nu-j)!(\lambda-\nu+j)!(\lambda-\mu-2\nu+2j)!}\binom{M}{r-i-j}. \tag{3.4}$$

Recalling that, when $\mu = 2M+1$, $l+\lambda$ should be an odd number for the integral in (1.1) not to vanish, just for the sake of clarity we explicitly write the result given above for the two possible cases.

(a) When $l=2L$ (l is even) and $\lambda = 2\Lambda+1$ (λ is odd) we have

$$
\begin{aligned}
I(l=2L, \lambda = 2\Lambda+1, \mu = 2M+1)&\\
=\frac{(-1)^{(\mu-\lambda-l)}\pi}{2^{l+\lambda+1}} &\sum_{r=0}^{(l+\lambda-1)/2} \frac{(-1)^{-r}(2r)!}{2^{2r}r!(r+1)!}\\
&\times \sum_{i=\max\{0,r-(\lambda-1)/2\}}^{\min\{r,l/2\}} \frac{(l+2i)!}{(l/2-i)!(l/2+i)!(2i)!}\\
&\times \sum_{j=\max\{0,r-i-(\mu-1)/2\}}^{\min\{r-i,(\lambda-\mu)/2\}} \binom{(\mu-1)/2}{r-i-j} \frac{(\lambda+\mu+2j)!}{[(\lambda-\mu)/2-j]![(\lambda+\mu)/2+j]!(2j)!}.
\end{aligned} \tag{3.5}
$$

(b) When $l=2L+1$ (l is odd) and $\lambda = 2\Lambda$ (λ is even) we have

$$
\begin{aligned}
I(l=2L+1, \lambda = 2\Lambda, \mu = 2M+1)&\\
=\frac{(-1)^{(\mu-\lambda-l)}\pi}{2^{l+\lambda+3}} &\sum_{r=0}^{(l+\lambda-3)/2} \frac{(-1)^{-r}(2r+2)!}{2^{2r}(r+1)!(r+2)!}\\
&\times \sum_{i=\max\{0,r-\lambda/2+1\}}^{\min\{r,(l-1)/2\}} \frac{(l+2i+1)!}{[(l-1)/2-i]![(l+1)/2+i]!(2i+1)!}\\
&\times \sum_{j=\max\{0,r-i-(\mu-1)/2\}}^{\min\{r-i,(\lambda-\mu-3)/2\}} \binom{(\mu-1)/2}{r-i-j}\\
&\times \frac{(\lambda+\mu+2j+3)!}{[(\lambda-\mu-3)/2-j]![(\lambda+\mu+3)/2+j]!(2j+3)!}.
\end{aligned} \tag{3.6}
$$

4. Summary

Although the results are somewhat involved we have obtained closed expressions for integrals of products of Legendre polynomials and their associated Legendre functions for all possible values of the $\{l, \lambda, \mu\}$ indices. In table 1 we indicate which equation should be used (or the result when it is null) in order to evaluate each integral, depending on the parity of the indices l, λ and μ.

Table 1. Result or equation to be used in evaluating $I(l, \lambda, \mu)$ according to the parity of each of the values of l, λ and μ.

l	λ	μ	$I(l, \lambda, \mu)$
+	+	+	(2.12)
+	+	−	0
+	−	+	0
+	−	−	(3.5)
−	+	+	0
−	+	−	(3.6)
−	−	+	(2.13)
−	−	−	0

Table 2. Non-null values obtained for $\tilde{I}(l, \lambda, \mu)$ when $1 \leq \mu = \lambda = 3$ and $0 < l < l_{max}$. The maximum value of l included, namely l_{max}, corresponds to cutoff in the sum rule of (4.2): $|1 - S_{\lambda\mu}(l_{max})| < 10^{-3}$.

λ	μ	l	$\tilde{I}(l, \lambda, \mu)$	$S_{\lambda\mu}(l)$
3	1	0	0.224 947	0.0506
		2	0.880 243	0.8254
		4	−0.405 958	0.9902
		6	−0.087 1252	0.9978
		8	−0.037 6449	0.9992
	2	1	0.836 66	0.7000
		3	−0.547 723	1.0000
	3	0	0.871 215	0.7590
		2	−0.487 024	0.9962
		4	0.061 257	0.9999

Table 3. As table 2 for $\lambda = 4$.

λ	μ	l	$\tilde{I}(l, \lambda, \mu)$	$S_{\lambda\mu}(l)$
4	1	1	−0.285 172	0.0813
		3	−0.843 989	0.7936
		5	−0.439 41	0.9867
		7	0.100 258	0.9968
		9	0.045 073	0.9988
	2	0	0.316 228	0.1000
		2	0.707 107	0.6000
		4	−0.632 456	1.0000
	3	1	−0.754 494	0.5693
		3	0.648 286	0.9895
		5	−0.101 584	0.9999
	4	0	0.836 66	0.7000
		2	−0.534 522	0.9857
		4	0.119 523	1.0000

To evaluate the expression so far obtained we have written a code (available on request) in the algebraic programming language SMP [6]. In order to show the general trend we present in tables 2–4, and for different values of the indices $\{l, \lambda, \mu\}$, the related quantity

$$\tilde{I}(l, \lambda, \mu) \equiv \left(\frac{(2l+1)}{2}\frac{(2\lambda+1)}{2}\frac{(\lambda-\mu)!}{(\lambda+\mu)!}\right)^{1/2} I(l, \lambda, \mu)$$

$$= \int_{-1}^{1} dx\, \tilde{P}_l(x) \tilde{P}_\lambda^\mu(x) \tag{4.1}$$

which can be interpreted as the lth coefficient in a (normalised) Legendre polynomial expansion of the (normalised) Legendre associated function:

$$\tilde{P}_l(x) \equiv \left(\frac{(2l+1)}{2}\right)^{1/2} P_l(x)$$

Table 4. As table 2 for $\lambda = 5$.

λ	μ	l	$\tilde{I}(l, \lambda, \mu)$	$S_{\lambda\mu}(l)$
5	1	0	0.111 465	0.0124
		2	0.295 975	0.1000
		4	0.817 697	0.7686
		6	−0.463 509	0.9835
		8	−0.110 547	0.9957
	2	1	0.396 412	0.1571
		3	0.605 53	0.5238
		5	−0.690 066	1.0000
	3	0	0.361 187	0.1305
		2	0.555 251	0.4388
		4	−0.736 482	0.9813
		6	0.136 078	0.9997
		8	0.0169 972	0.9999
	4	1	0.686 607	0.4714
		3	−0.699 206	0.9603
		5	0.199 205	1.0000
	5	0	0.807 638	0.6523
		2	−0.564 354	0.9708
		4	0.170 361	0.9998
		6	−0.0142 186	0.9999

$$\tilde{P}^{\mu}_{\lambda}(x) \equiv \left(\frac{(2\lambda+1)}{2}\frac{(\lambda-|\mu|)!}{(\lambda+|\mu|)!}\right)^{1/2} P^{\mu}_{\lambda}(x)$$

where the tilde was included to indicate normalisation. Because of norm conservation, we get a condition on the sum:

$$S_{\lambda\mu}(l) \equiv \sum_{l'=0}^{l} |\tilde{I}(l', \lambda, \mu)|^2 \xrightarrow{l\to\infty} 1. \tag{4.2}$$

This result is also included in tables 2–4. For μ even, the sum runs up to $l = \lambda$, as higher l' contributions vanish (see selection rule (1.6)). On the other hand, for μ odd, (4.2) is extremely useful for estimating a cutoff point beyond which the $I(l, \lambda, \mu)$ may be neglected for a given approximation. We do not include in tables 2–4 those cases for which the integral vanishes according to the selection rules derived in § 2.

As a final remark we note that the same approach outlined above can be used to evaluate integrals involving two associated Legendre functions, a result which is also missing in the literature, and which should (perhaps) be looked at for the sake of completeness. Nevertheless, we have not done it here as it was not relevant to the study of excitation modes in deformed systems, which was our primary interest.

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